HEAT TRANSFER IN LAMINAR FLOW. III.*
APPLICATION OF THE THERMAL BOUNDARY LAYER

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The concept of the thermal boundary layer in the entry region of a heat exchanger with developed velocity profile serves to calculate in an approximate manner the temperature field. Several approaches to the solution of the thermal boundary layer are discussed.

In the papers[1,2] we have dealt with the problem of heat transfer in an annular, circular or a flat duct, on the walls of which a temperature step is imposed (Fig. 1). In dimensionless form this problem (the Grätz–Nusselt type) may be stated as follows

$$w(y)\frac{\partial t}{\partial z} = \frac{\partial^2 t}{\partial y^2} - \frac{1 - \varkappa}{1 + y(1 - \varkappa)}\frac{\partial t}{\partial y}, \tag{1}$$

with the boundary conditions

$$t = 1 \quad \text{for} \quad z \leqq 0\,, \quad t = 0 \quad \text{for} \quad y = 0 \quad \text{and} \quad z \geqq 0\,, \qquad (2), (3)$$

$$\frac{\partial t}{\partial y} = 0 \quad \text{for} \quad y = 1\,. \tag{4}$$

Our attention was focused on the question of the amount of heat transferred in the exchanger. However, there are certain arrangements in which the heat transfer is described by the same equation and the boundary conditions but, in addition to the heat transfer, there is another process going on (such as *e.g.* chemical reaction) which is strongly affected by temperature. Then the need arises for the knowledge of the thermal field and, simultaneously, the problem of how to describe the temperature distribution. By exact solution of the set[1] (*1*)–(*4*) we obtained the relation

$$t(y, z) = \sum_{i=1}^{\infty} c_i Y_i(y) \exp(-b_i^2 z)\,, \tag{5}$$

which is unfortunately rather awkward at low values of z owing to slow convergence of the series. The difficulties do not concern the evaluation of the series proper but rather the neccesity to tabulate a great number of functions and constants.

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In this paper an approximate method of solving the heat transfer equation will be presented which enables the temperature distribution to be described in a simple and illustrative manner without undue loss of accuracy.

THERMAL BOUNDARY LAYER

The equation of heat transfer without the term for axial conduction (*1*) possesses a similar structure as the Prandtl equation for a twodimensional velocity boundary layer[3]. There is no feed-back in axial direction and thus the step change in boundary conditions results also in formation of the boundary layer into which the response to the change is localized. This fact may be demonstrated by the example of the temperature field calculated from Eq. (*5*) for a newtonian flow through a pipe when $\varkappa = 0$ and $w(y) = 4y - 2y^2$, shown in Fig. 2. It is seen that for small z there exists region practically unaffected by the change of wall temperature and a region of intensive heat transfer – the region of the boundary layer. It should be noted that this is not specific for the selected example; qualitatively same results were obtained by sol-

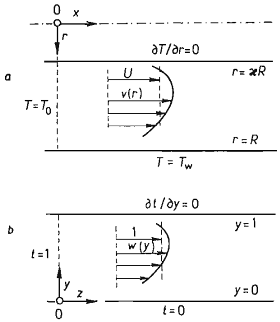

FIG. 1

Heat Transfer Problem under Discussion

a) Sketch of heat transfer in an annular duct. On the wall of radius R the temperature jumps from T_0 to T_w; the wall of the radius $\varkappa R$ is insulated. *b*) Dimensionless formulation of the problem. Into the same form transform the cases: $\varkappa = 0$ (tube), $\varkappa = 1$ (flat duct with heat exchange on one wall or both) and $\varkappa > 1$ (annular duct with heat exchange on the inner surface).

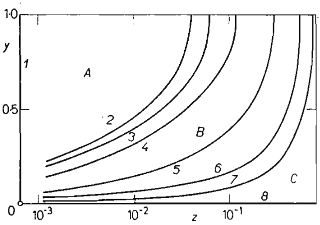

FIG. 2

Temperature Field in a Tubular Heat Exchanger with Constant Wall Temperature under Newtonian Type of Flow

Isotherms belong to following dimensionless temperatures: t: *1* 1; *2* 0·999; *3* 0·99; *4* 0·9; *5* 0·5; *6* 0·2; *7* 0·1; *8* 0. Region: *A* unaffected by wall temperature, *B* of marked heat transfer (boundary layer); *C* of vanishing temperature profile.

ving[4] the Grätz–Nusselt problem for a variety of velocity profiles $w(y)$, and geometric parameters $\varkappa$. Thus we are justified to state that: To every arbitrarily small ε there exists a nondecreasing function $y_0 = y_0(z, \varepsilon)$ defined in the domain $0 \leqq z \leqq z_e(\varepsilon)$ and such that for $y = y_0(z, \varepsilon)$ $t(y, z) = 1 - \varepsilon$. Thus we assume that the lateral temperature profile, $t(y)$, may be approximated, for given $z \leqq z_e$, by two functions:
– for $y \leqq y_0(z)$ by function $t(y)$ of which we only know so far that it should satisfy the following conditions

$$t < 1 \quad \text{and} \quad \partial t/\partial y > 0 \quad \text{for} \quad 0 \leqq y < y_0\,, \tag{6}$$

$$t = 1 \quad \text{and} \quad \partial t/\partial y = 0 \quad \text{for} \quad y = y_0\,, \tag{7}$$

$$t = 0 \qquad\qquad \text{for} \quad y = 0\,; \tag{3}$$

for $y_0 \leqq y \leqq 1$ by function $\quad t = 1\,.$ (8)

Adopting this approximation it remains to solve the problem of the boundary layer, *i.e.* to find functions $y_0(z)$ and $t(y, z)$ (for $y \leqq y_0$) which satisfy best the original set of Eqs (*1*)–(*4*). Additional (derived) results of the solution are the length of the inlet region z_e (which is the minimum value of z for which $y_0 = 1$) and the average mixing temperature

$$t_m(z) = 2/(1 + \varkappa) \left\{\int_0^{y_0(z)} t(y, z)\, w(y) \left[1 - y(1 - \varkappa)\right] \mathrm{d}y + \right.$$

$$\left. + \int_{y_0(z)}^1 w(y) \left[1 - y(1 - \varkappa)\right] \mathrm{d}y \right\}, \tag{9}$$

or other integral characteristics of heat transfer[2].

Solution of the Equation for the Thermal Boundary Layer

The known methods for solution of the velocity profiles in the vicinity of submerged bodies[3] may be used also for the thermal boundary layer. The simplest is the one assuming similarity of the temperature profiles in the boundary layer

$$t(y, z) = t(\eta)\,, \quad \text{where} \quad \eta = y/y_0(z)\,. \tag{10), (11}$$

On taking an arbitrary function $t(\eta)$ which satisfies conditions (*3*), (*6*) and (*7*), *i.e.* $t(0) = 0$, $t'(1) = 0$ and $t'(\eta) > 0$ for $\eta < 1$ we can calculate the axial coordinate corresponding to a given thickness of the thermal boundary layer from the relation

$$z = -(1 + \varkappa)/[2t'(0)] \int_0^{y_0(z)} (\mathrm{d}t_m/\mathrm{d}y_0) y_0 \, \mathrm{d}y_0 \qquad (12)$$

expressing the balance on heat. Since the functions $t_m(y_0)$ and $z(y_0)$ are inversible, the required result, *i.e.* the description of the thermal field, may also be obtained by means of functions $y_0(z)$, $t(\eta)$ and $t_m(z)$. The methods of the variation calculus based on the theorem[5] of minimum entropy production may be used eventually for the calculation of $z(y_0)$ instead of Eq. (*12*). An improved method of the thermal boundary layer, which shall be described in the following, differs from previous methods in that it does not require any assumptions regarding the similarity of temperature profiles. It utilises the fact that the set of Eqs (*1*) together with the conditions (*2*), (*3*) and (*8*) in the domain $0 \leqq y \leqq y_0(z_1)$; $0 \leqq z \leqq z_1 \leqq z_e$ (in Fig. 3 shadowed) forms again the Grätz–Nusselt type of problem. From the viewpoint of thus formulated the Grätz–Nusselt problem the entry region is terminated in the point z_1 and, consequently, z_1 has here the same meaning as z_e in the original problem. For $z = z_1$ it is therefore possible to approximate the temperature profile by the first eigenfunction of the Sturm–Liouville problem

$$Y_1'' + (1 - \varkappa)/[1 - y(1 - \varkappa)] \, Y_1' - w(y) \, b_1^2 Y_1 = 0 \,, \qquad (13)$$

where b_1^2 is the smallest eigenvalue satisfying both Eq. (*13*) and conditions

$$Y_1 = 0 \quad \text{for} \quad y = 0 \,, \quad Y_1' = 0 \quad \text{and} \quad Y_1 = 1 \quad \text{for} \quad y = y_0 \,. \qquad (14), (15)$$

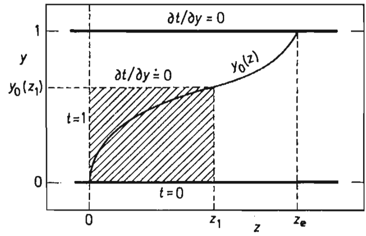

FIG. 3

Entry Region of Heat Exchanger

Broken line shows the limits of the boundary layer. The boundary conditions in the shadowed area are those of the Grätz–Nusselt problem.

Obviously, Y_1 is a function of a single variable, y, and depends, similarly as b_1^2, on parameter y_0. The temperature in the boundary layer is approximated by the relation

$$t(y) = Y_1(y)\,. \tag{16}$$

If the function $Y(y)$ and the value b_1^2 for a given y_0 are found one can calculate the appropriate coordinate z_1 on the basis of relations equivalent to Eq. (5) as

$$z_1 = \ln c_1/b_1^2\,, \tag{17}$$

making use of the following relations:

$$c_1 = 1/A \int_0^{y_0} [1 - y(1 - \varkappa)]\, w(y)\, Y_1(y)\, \mathrm{d}y\,, \tag{18}$$

$$b_1^2 = 1/A \int_0^{y_0} [1 - y(1 - \varkappa)]\, Y_1'^2(y)\, \mathrm{d}y\,, \tag{19}$$

where

$$A = \int_0^{y_0} [1 - y(1 - \varkappa)]\, w(y)\, Y_1^2(y)\, \mathrm{d}y\,. \tag{20}$$

The functions $y_0(z)$, $t(z, y)$ for $y \leqq y_0$ *etc.* can be obtained again by appropriate inversion of known functions. The knowledge of the function $Y_1(y)$ and the values

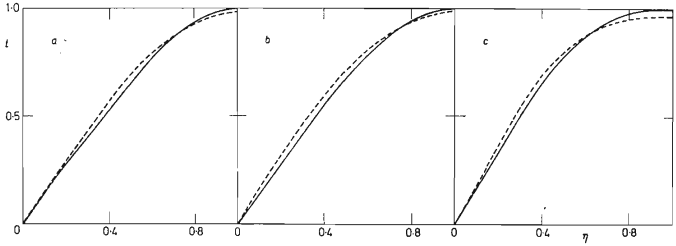

FIG. 4

Thermal Boundary Layer at Heat Transfer into a Newtonian Liquid ($w(y) = 6y - 6y^2$) in a Flat Duct with Temperature Step on One Wall

Solid line shows the temperature profile calculated assuming the boundary layer; broken line is the exact solution. *a*) Limiting case for the neighbourhood of entrance of the exchanger ($z \to 0$). The thickness of the boundary layer is $y_0 = 1{\cdot}527z^{1/3}$. The exact solution is given by Eq. (*28*); *b* $z_1 = 0{\cdot}0248$; $y_0(z_1) = 0{\cdot}5$; *c* $z_1 = 0{\cdot}0912$; $y_0(z_1) = 1$.

c_1 and b_1^2 for $y_0 = 1$ permits description of the temperature field for $z \geqq z_e$. An approximation of the series (5) by its first term is very accurate in this region[2].

A solution of the Grätz–Nusselt problem given by Eq. (1) and the boundary conditions (2) to (4) may be written in the form (5). Since[2] $b_{i+1}^2 < b_i^2$ and $c_{i+1} < c_i$ we have that

$$\lim_{z \to \infty} t(y, z) = c_1 Y_1(y) \exp(-b_1^2 z) . \qquad (21)$$

If the relation (21) is extrapolated toward smaller values of z one obtains meaningless results $t(1, z) < 1$ if $z < \ln c_1/b_1^2$. In the point $z_e = \ln c_1/b_1^2$ we obtain by extrapolation that

$$t(y, z_e) = Y_1(y) . \qquad (22)$$

The temperature calculated from Eq. (22) deviates from the exact solution (5) in this point by

$$\Delta t = Y_1(y) - t(y, z_e) = - \sum_{i=2}^{\infty} c_i Y_i(y) c_1^{-b_i^2/b_1^2} \qquad (23)$$

and the deviation reaches maximum at the point $y = 1$

$$\Delta t_s = - \sum_{i=2}^{\infty} c_i c_1^{-b_i^2/b_1^2} . \qquad (24)$$

This deviation was evaluated numerically for a number of geometric arrangements and velocity profiles. The maximum deviation was found in the case of piston flow in a pipe $\Delta t_s = -0.088$. In the case of real newtonian or non-Newtonian flows we cannot commit an error greater than 5% by using Eq. (22) instead of (5) and the error of determination of the average mixing temperature does not even exceed 1%.

Calculation of the First Eigenfunction

The methods for calculation of the eigenfunctions are ample and can be divided into four following groups: *1*. In some cases the function $Y_1(y)$ can be expressed in terms of tabulated functions. In the case of the piston flow in a flat duct, $\varkappa = 1$, $w(y) = 1$, it is an elementary function

$$Y_1 = \sin \{[\ln (\pi/4)]^{1/2} y/(z)^{1/2}\} , \qquad (25)$$

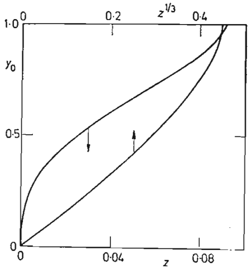

FIG. 5

The Plot of Function $y_0(z_1)$ for Given Case (Fig. 4) which may serve for comparison *e.g.* the result of integration in Eq. (*9*) $t_m = 1 - 1{\cdot}161(z)^{1/2}$ for $z \leqq 0{\cdot}0979$, with the result of the exact method[6] $t_m = 1 - 1{\cdot}128(z)^{1/2}$ for $z \to 0$. In other examples of the piston flow $Y_1(y)$ can be found as a combination of the Bessel functions[6]. A noteworthy situation arises with a linear dependence $w(y) = w'_0 y$ in the boundary layer for which we obtain[4]

$$Y_1 = 0{\cdot}856\zeta^{1/2}J_{1/3}(0{\cdot}2687\zeta^{3/2}), \quad \text{for} \quad \zeta \leqq 2{\cdot}776 \qquad (26)$$

where

$$\zeta = y(w'_0)^{1/3}/z^{1/3}. \qquad (27)$$

Relation (*26*) may be compared with the exact solution of Leveque[7,8]

$$t = 1{\cdot}120 \int_0^{0{\cdot}481\,\zeta} \exp(-\xi^3)\,d\xi, \quad \text{for} \quad \zeta \to 0. \qquad (28)$$

The courses of the functions (*26*) and (*27*) are shown in Fig. 4 plotting t in dependence on $\eta = \zeta/2{\cdot}776$. *2*. The method used almost exclusively in pre-computer era[9,10], which rests in expanding the functions $w(y)$ and $Y_1(y)$ into a polynomial. On comparing the terms with the same power of y in Eq. (*13*) a set of linear equations is obtained the roots of which are the coefficients of polynomials of the eigenfunctions and the eigenvalues. The method is suitable for the calculation of the first eigenfunction providing that the velocity profile is expressed by a finite polynomial, although even then the eigenfunction must be described by a polynomial of about 10 terms to achieve accuracy of 1%. Thus here too it is convenient to use some standard (matrix inversion) computer routine. *3*. The method based on the variation calculus, which usually utilize the fact that the solution $Y_1(y)$ of the Sturm–Liouville set of Eqs (*13*)—(*15*) is on the extremal of the functional

$$I(Y'_1, Y_1, y) = \int_0^{y_0} \{[1 - y(1 - \varkappa)]\, Y_1'^2 - b_1^2 w(y)\,[1 - y(1 - \varkappa)]\, Y_1^2\}\,dy, \qquad (29)$$

where b_1^2 is the functional given by Eqs (*19*) and (*20*). To find the extremal

$$\delta I = 0 \qquad (30)$$

one can use for instance Ritz's method consisting in expressing $Y_1(y)$ as a series of functions $F_i(y)$

$$Y_1(y) = a_1F_1(y) + a_2F_2(y) + \ldots, \qquad (31)$$

which all satisfy the boundary conditions

$$F_i(0) = 0, \quad F'_i(y_0) = 0. \qquad (32), (33)$$

If Eq. (*30*) is to hold such values a_i must be found for which

$$\partial I/\partial a_i = 0. \qquad (34)$$

To diminish the number of functions $F_i(y)$ necessary for a satisfactory convergence of the series (*31*) it is convenient to choose the orthogonal functions with the weight $w(y)\,[1 - y(1 - \varkappa)]$. This means that we request that

$$\int_0^{y_0} F_i(y)\, F_j(y)\, w(y)\, [1 - y(1 - \varkappa)]\, dy = 0\,, \quad \text{for} \quad i = j\,. \tag{35}$$

A combination of two such orthogonal functions (*31*) with one and two local extremes in the domain $0 \leqq y \leqq y_0$ usually suffice[4,11] for a requested accuracy of determination of $Y_1(y)$. If the velocity profile is described by a polynomial, these functions may also be taken as polynomials and for the solution of the set of Eqs (*34*) we then do with the simplest mathematical means. Quite generally it may be stated that the approximate approaches based on the variation calculus lead very rapidly to reliable results. *4.* For a numerical solution it is convenient to avoid the Sturm-Liouville boundary value problem and to change it into the Cauchy initial value problem. In such a case we integrate the differential equation (*13*) with the initial condition $Y(0) = 0$ and arbitrarily selected values b_1^2 and $Y'(0)$ until reaching the point $y = y_0$ in which the first local extreme $Y_0' = 0$ is found. (If the value $y = 1$ was exceeded we had taken b_1^2 too small). If $y_0 \leqq 1$, the appropriate first eigenfunction is given by the relation

$$Y_1(y) = Y(y)/Y(y_0) \tag{36}$$

and the appropriate coordinate z is determined with the aid of Eqs (*17*)—(*20*). The presented method enables the first eigenfunction to be determined even in cases when the velocity profile is not described by a simple function. The concept of the boundary layer enables to describe in an illustrative manner the temperature pattern in the entry region of a heat exchanger (Figs 4, 5). The results of the heat transfer in ducts with a single heat exchanging wall, which are solutions of the set of Eqs (*1*)—(*4*), may be used for description of the temperature field in the entry region with heat exchange on two walls since both boundary layers will not affect each other initially. Also other examples (*e.g.* when thermal fluxes on the walls are specified may be modified by the above presented method. The assumption of the boundary layer provides also an approach to the solution of heat transfer in a liquid with temperature dependent viscosity[12].

LIST OF SYMBOLS

Dimensional quantities

c_p	specific heat
k	thermal conductivity
r	radial coordinate
R	radius of the wall with temperature step
T	temperature
T_0	inlet temperature of liquid
T_w	temperature of heat exchanging surface
U	average velocity of liquid
v	local velocity of liquid
x	axial coordinate
ϱ	density

Dimensionless quantities

a_i	coefficient of series (*20*)
A	functional defined by Eq. (*20*)
b_1^2	eigenvalues

c_i coefficient of series (*5*)
F function satisfying boundary conditions (*32*) and (*33*)
t $= (T - T_w)/(T_0 - T_w)$ temperature
t_m mean mixing temperature
Δt error caused by approximation of temperature profile (*23*)
Δt_s maximum Δt
w $= v/U$ velocity
w'_0 $= dw/dy$ at the point $y = 0$; velocity gradient on the wall
y $= (R - r)/(R - \varkappa R)$ lateral coordinate
y_0 thickness of the thermal boundary layer
Y auxiliary function satisfying Eq. (*13*)
Y_i eingenfunctions
$Y'_1 Y''_1$ derivatives of Y_i with respect to y
z $= xk/(\varrho c_p UR^2(1 - \varkappa^2))$ axial coordinate
z_e length of entry region
z_1 selected value $z \leqq z_e$
δ variation
ζ variable defined by Eq. (*27*)
η normalized variable for the boundary layer (*11*)
$\varkappa$ geometric parameter (Fig. 1)

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